MELLIN TRANSFORM APPLIED TO UNCERTAINTY ANALYSIS IN HYDROLOGY/HYDRAULICS

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Mellin Transform Applied to Uncertainty Analysis in Hydrology/Hydraulics

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Abstract: In hydraulic and hydrologic design and analysis, engineers frequently face uncertainties involving quantities that cannot be assessed with absolute accuracy. Under such circumstances, uncertainty analysis is undertaken to examine the effects of uncertain factors on the results of design and analysis. The paper introduces a mathematical technique called the Mellin transform. The technique is analytically convenient in determining the exact statistical moments of a random variable that is a function of several nonnegative independent random variables in a multiplicative form. Two examples are given to demonstrate the application of the Mellin transform to uncertainty analysis of hydrologic and hydraulic problems.

Introduction

In hydrologic and hydraulic analyses, engineers frequently encounter quantities that cannot be quantified with certainty. The existence of uncertainties directly affects the performance reliability of the hydraulic structure being designed.

Uncertainties in hydrologic and hydraulic modeling can broadly be classified into two types (Tung and Mays 1980): model uncertainty and parameter uncertainty. Hydrologic and hydraulic designs invariably involve the use of equations that are empirically developed or analytically derived under some idealized conditions. Model uncertainty results from the use of a simplified equation to describe a complex hydraulic or hydrologic flow phenomenon and flow process. For example, Manning's equation and other steady-state uniform-flow equations are commonly used in open-channel analysis; the rational formula and different forms of equations for the time of concentration are used in urban drainage-structure design.

All hydrologic and hydraulic equations involve several physical parameters that cannot be quantified accurately. This is the parameter uncertainty. Parameter uncertainty could be caused by change in operational conditions of hydraulic structures, inherent variability of input parameters in time and in space, and lack of a sufficient amount of data. Consequently, hydrologic and hydraulic quantities such as the average flow velocity in the channel and peak discharge of urban runoff cannot be assessed with certainty.

The main objective of uncertainty analysis is to identify the statistical properties of a model output as the function of stochastic input parameters. This paper describes a mathematical integral transform technique called the Mellin transform and demonstrates its applications to the uncertainty analysis of hydrologic and hydraulic problems.

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Uncertainty Analysis

In hydrologic and hydraulic analyses, models or equations that relate a dependent variable Y (e.g., peak discharge, flow velocity, time of concentration, etc.) to a number of model parameters $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ (e.g., roughness factor, channel geometry and slope, etc.) can be expressed

$$Y = t(\mathbf{X}) = t(X_1, X_2, \ldots, X_k) \quad (1)$$

In cases where some of the model input parameters are subject to uncertainty, the value that the dependent variable Y takes is uncertain. Uncertainty characteristics of a hydrologic or hydraulic variable under consideration can be defined by its distribution and statistical moments such as the mean, variance, coefficient of variation, and other higher moments, if necessary. Ideally, in uncertainty analysis, derivation of the exact probability density function (PDF) of Y as the function of the PDFs of stochastic input parameters X in the model is desirable. However, such a task is difficult, if not impossible, because of the nonlinearity in the model. In most engineering designs and analyses, it is generally sufficient to estimate the first few statistical moments of Y as the function of the statistical moments of stochastic input parameters.

One commonly used method for assessing the statistical moments in the uncertainty analysis is the mean value first-order second-moment (MFOSM) method (Benjamin and Cornell 1970; Ang and Tang 1975). Recently, Yen et al. (1986) gave a very comprehensive evaluation and description of the MFOSM method in uncertainty and risk analyses. Note that the MFOSM method only gives approximations of the mean and variance, instead of their exact values. When higher-order moments are needed, the MFOSM method becomes computationally cumbersome as the order gets larger. Furthermore, evidence shows that as the nonlinearity of the functional relation gets higher, the accuracy of approximation by the MFOSM method deteriorates rapidly, especially for high-order moments (Gardner et al. 1981; Tung and Hathhorn 1988).

The other type of method useful in uncertainty analysis is the integral transform techniques. Some well-known integral transforms are the Fourier, Laplace, and exponential transforms. The present paper describes a transform technique called the Mellin transform (Epstein 1948; Park 1987), which is less known to the hydraulics engineering community, and shows its applications.

If the functional relation of Eq. 1 satisfies two conditions, the exact moments of any order can be derived analytically as the function of moments of stochastic input parameters X by the Mellin transform without extensive simulation or using approximation by MFOSM method. The two conditions are: (1) The function $t(\mathbf{X})$ has a multiplicative form

$$Y = t(\mathbf{X}) = a_0 \prod_{i=1}^{k} X^{a_i} \quad (2)$$

where a_i are constants; and (2) the stochastic input parameters, Xs, are independent and nonnegative. The Mellin transform is particularly attractive in uncertainty analysis of hydrologic and hydraulic problems because many equations and their parameters involved satisfy these two conditions.

In general, the nonnegativity condition of the Xs is not strictly required in the Mellin transform; but it would require some mathematical manipulations to find the Mellin transform of a function in which random variables can take negative values (Epstein 1948; Springer 1978).

MOMENTS AND MELLIN TRANSFORM

In this section, the definitions of statistical moments of a random variable and Mellin transform are given to show the relationships between the two.

Statistical Moments

The statistical moment of order r of a random variable X about a reference point $X = x_0$ is defined as

$$\mu_r^0 = E[(X - x_0)^r] = \int_{-\infty}^{\infty} (x - x_0)^r f(x)dx \quad \text{(3)}$$

where $E[\ ]$ = an expectation operator; and $f(x)$ = PDF of random variable X. In general, statistical moments in uncertainty analysis that are commonly used are central moments with reference point $x_0 = \mu$ and the moments about the origin with $x_0 = 0$. In other words, the central moments and moments about origin can be defined, respectively, as

$$\mu_r = E[(X - \mu)^r] \quad \text{(4)}$$

$$\mu_r' = E[X^r] \quad \text{(5)}$$

where $\mu = E[X]$ = expectation of random variable X. It can be easily shown, through the binomial expansion, that the central moments can be obtained from the moments about the origin as

$$\mu_r = \sum_{i=0}^{r} {}_rC_i(-1)^i\mu^i\mu_{r-i}' \quad \text{(6)}$$

where ${}_rC_i = r!/[(r - i)!i!]$. More specifically, the second, third, and fourth central moments can be expressed as

$$\sigma^2 = E(X^2) - \mu^2 \quad \text{(7)}$$

$$\mu_3 = E(X^3) - 3\mu E(X^2) + 2\mu^3 \quad \text{(8)}$$

$$\mu_4 = E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4 \quad \text{(9)}$$

in which σ^2 = variance. From Eqs. 7–9, one can calculate the skew coefficient (γ) and kurtosis (κ) as

$$\gamma = \frac{\mu_3}{\sigma^3} \quad \text{(10)}$$

$$\kappa = \frac{\mu_4}{\sigma^4} \quad \text{(11)}$$

Mellin Transform

The Mellin transform of a function $f(x)$, where x is positive, is defined as (Giffin 1975; Springer 1978)

$$M_x(s) = M[f(x)] = \int_0^{\infty} x^{s-1}f(x)dx, \qquad x > 0 \quad \text{(12)}$$

where $M_x(s)$ is the Mellin transform of $f(x)$. As in the Fourier and Laplace transforms, there exists a one-to-one correspondence between $M_x(s)$ and $f(x)$. When $f(x)$ is a PDF, one can immediately recognize the relationship between the Mellin transform of a PDF and the moments about the origin as

$$\mu_{s-1}' = E(X^{s-1}) = M_x(s) \quad \text{(13)}$$

for $s = 1, 2, \ldots$. As can be seen, the Mellin transform provides an alternative way to find the moments of any order of a nonnegative random variable.

OPERATIONAL PROPERTIES OF MELLIN TRANSFORM

Consider that a random variable Z is the product of two independent nonnegative random variables, i.e. $Z = XY$. The PDF of Z, $f(z)$, can be obtained as

$$f(z) = \int_0^{\infty} g\left(\frac{z}{y}\right) h(y)dy \quad \text{(14)}$$

where $g(\ )$ and $h(\ )$ = PDF's of X and Y, respectively. In fact, Eq. 14 is exactly the definition of Mellin convolution (Springer 1978). Therefore, similar to the convolutional property of the Laplace and Fourier transforms, the Mellin transform of $f(z)$ can be obtained as

$$M_z(s) = M[f(x)] = M[g(x)^*h(y)] = M_x(s)M_y(s) \quad \text{(15)}$$

in which * = convolution operator. From Eq. 15, the Mellin transform of the convolution of the PDFs associated with two independent random variables in a product form simply is equal to the product of the Mellin transform of two individual PDFs. Eq. 15 can be extended to a general case involving more than two independent random variables.

From this convolutional property of the Mellin transform and its relationship between statistical moments, one can immediately see the advantage of the Mellin transform as a tool in obtaining the moments of a random variable that is related to other random variables in a multiplicative fashion. In addition to the convolutional property, which is of primary importance, the Mellin transform also has several useful operational properties, which are summarized in Table 1 (Bateman 1954; Park 1987). These properties of the

TABLE 1. Operation Properties of Mellin Transform on a PDF

Property (1)	PDF (2)	Random variable (3)	Mellin transform (4)
Standard	$f(x)$	X	$M_x(s)$
Scaling	$f(ax)$	X	$a^{-s}M_x(s)$
Linear	$af(x)$	X	$aM_x(s)$
Translation	$x^af(x)$	X	$M_x(a + s)$
Exponentiation	$f(x^a)$	X	$a^{-1}M_x(s/a)$

TABLE 2. Mellin Transform of Products and Quotients of Random Variables

Random variable (1)	PDF given (2)	$M_z(s) =$ (3)
$Z = X$	$f(x)$	$M_x(s)$
$Z = X^b$	$f(x)$	$M_x(bs - b + 1)$
$Z = 1/X$	$f(x)$	$M_x(2 - s)$
$Z = XY$	$f(x)$, $g(y)$	$M_x(s)M_y(s)$
$Z = X/Y$	$f(x)$, $g(y)$	$M_x(s)M_y(2 - s)$
$Z = aX^bY^c$	$f(x)$, $g(y)$	$a^{s-1}M_x(bs - b + 1)M_y(cs - c + 1)$

Note: a, b, c = constants; and X, Y, Z = random variables.

Mellin transform can be derived from the basic definition given in Eq. 12.

Applying the definition of the Mellin transform and its basic operational properties, along with the convolutional properties, the Mellin transform of algebra of random variables in the form of products and quotients can be derived. Some useful results are summarized in Table 2 (Park 1987).

Mellin Transform of Some Probability Density Functions

In uncertainty analysis, model parameters with uncertainty are treated as random variables associated with a PDF. Given the functional relationship as Eq. 2, in which Y is related to the Xs in a multiplicative fashion, the statistical moments of Y can be obtained by Mellin transform of the PDF of the Xs. From previous studies (Epstein 1948; Park 1987), the Mellin transform of some commonly used PDFs are tabulated in Table 3. Using the results in Tables 2 and 3, one can derive the exact moments of the dependent random variable Y.

Although the Mellin transform is useful for uncertainty analysis under the conditions stated previously, it possesses one drawback that should be pointed out: under some certain combinations of distribution and functional form, the resulting transform may not be analytic for all ss. This could occur especially when quotients or variables with negative exponents are involved. For example, if the random variable Y is related to the inverse of X, i.e. $Y = 1/X$, and X has a uniform distribution in (0,1), then $M_y(s) = M_x(2 - s) = 1/(2 - s)$. In this case, the expected value of Y, $E(Y)$, which can be calculated, in theory, by $M_y(s = 2)$, does not exist because $M_y(s = 2) = 1/0$, which is not defined. Under such circumstances, other transforms such as the Laplace transform could be used to find the moments.

Sensitivity of Component Uncertainty on Overall Uncertainty

In engineering designs, sensitivity analysis is commonly used when the designs are performed under uncertainty. In uncertainty analysis, investigating the impact of component uncertainty on the uncertainty level of the output provides important information regarding the relative contribution of component uncertainty to the overall uncertainty in model output.

Refer to a multiplicative model involving independent nonnegative random variables as in Eq. 2. The first two moments about the origin of the model output Y, using Table 2, can be obtained, respectively, as

TABLE 3. Mellin Transforms for Some Commonly Used Probability Density Functions

Probability (1)	$f(x)$ (2)	Mellin transform (3)
Uniform	$1/(b - a)$; $a \le x \le b$	$(b^s - a^s)/[s(b - a)]$
Exponential	ae^{-ax}; $x > 0$	$a^{1-s}\Gamma(s)$
Gamma	$[b(bx)^{a-1}e^{-bx}]/[\Gamma(a)]$; $x > 0$, $x > 0$, $b > 0$	$b^{1-s}[\Gamma(a + s - 1)]/[\Gamma(a)]$
Triangular	$[2(x - L)]/[(H - L)(M - L)]$; $0 \le L \le x \le M$ $[2(H - x)]/[(H - L)(H - M)]$; $M \le x \le H$	$2/[(H - L)s(s + 1)]\{[H(H^s - M^s)]/(H - M) - [L(M^s - L^s)]/(M - L)\}$
Standard beta	$[\Gamma(a + b)]/[\Gamma(a)\Gamma(b)]$; $0 \le x \le 1$, $a > 0$, $b > 0$	$[\Gamma(a + b)\Gamma(a + s - 1)]/[\Gamma(a)\Gamma(a + b + s - 1)]$
Nonstandard beta	$[\Gamma(a + b)]/[\Gamma(a)\Gamma(b)(H - L)](x - L)^{a-1}(H - x)^{b-1}$; $0 \le L \le x \le H$	$\Sigma_{k=0}^{s-1} \binom{s-1}{k} L^{s-1-k}(H - L)^k M_x(k + 1)$ where $M_x(k)$ for standard beta.
Normal	$1/(\sqrt{2\pi}) \exp(-x^2/2)$; $-\infty < x < \infty$	$(2)^{(s-3)/2}\Gamma(s/2)$

$$E(Y) = M_Y(2) = a_0 \prod_{i=1}^{k} M_{Xi}(1 + a_i) \quad (16)$$

$$E(Y^2) = M_Y(3) = a_0^2 \prod_{i=1}^{k} M_{Xi}(1 + 2a_i) \quad (17)$$

where $M_{Xi}(1 + a_i)$ and $M_{Xi}(1 + 2a_i)$ = the first two moments about the origin for the ith term, $Y_i = X_i^{a_i}$, in Eq. 2. The variance (Var) of the model output Y can be expressed as

$$\mathrm{Var}(Y) = a_0^2\left[\prod_{i=1}^{k} M_{Xi}(1 + 2a_i) - \prod_{i=1}^{k} M_{Xi}^2(1 + a_i)\right] \quad (18)$$

and the coefficient of variation (CV) as

$$CV^2(Y) + 1 = \prod_{i=1}^{k} [CV^2(Y_i) + 1] \quad (19)$$

where $CV(Y_i)$ is the coefficient of variation of the ith term in Eq. 2, i.e., $Y_i = X_i^{a_i}$.

To examine the impact of component uncertainty on the overall uncertainty in model output, it is necessary to express the coefficient of variation of Y in terms of the coefficients of variation of stochastic input parameters, X. Since $Y_i = X_i^{a_i}$, the relation between the coefficients of variation of Y_i and X_i can be similarly derived as

$$CV^2(Y_i) = \beta_i^2 CV^2(X_i) \quad (20)$$

where

$$\beta_i^2 = \left(\frac{M_{Xi}^2(2)}{M_{Xi}^2(1 + a_i)}\right)\left(\frac{M_{Xi}(1 + 2a_i) - M_{Xi}^2(1 + a_i)}{M_{Xi}(3) - M_{Xi}^2(2)}\right) \quad (21)$$

and $CV(X_i)$ = coefficient of variation of stochastic input parameter X_i which is computed as

$$CV(X_i) = \frac{\sqrt{M_{Xi}(3) - M_{Xi}^2(2)}}{M_{Xi}(2)} \quad (22)$$

Substituting Eq. 20 into Eq. 19 one obtains the following relation:

$$CV^2(Y) + 1 = \prod_{i=1}^{k} [\beta_i^2 CV^2(X_i) + 1] \quad (23)$$

The sensitivity of the model-output uncertainty with respect to the uncertainty of the ith stochastic parameter, X_i, can be obtained as

$$\frac{\partial CV(Y)}{\partial CV(X_i)} = \frac{\beta_i^2 CV(X_i)[CV^2(Y) + 1]}{CV(Y)[\beta_i^2 CV^2(X_i) + 1]} \quad (24)$$

The sensitivity coefficients computed by Eq. 24 represent the rate of change in model-output uncertainty resulting from a unit change in the ith input variable. Such information could be used as an important guide for future data-collection-program design in an attempt to reduce the total model-output uncertainty.

Referring to Eq. 23, it is seen that under Eq. 2, the relation between output uncertainty and the uncertainties of the input parameters is essentially multiplicative. Therefore, isolation of the exact impact of individual-component uncertainty is difficult. As an approximation, the MFOSM method can be applied, which leads to the following expression:

$$CV^2(Y) \approx \sum_{i=1}^{k} a_i^2 CV^2(X_i) \quad (25)$$

From Eq. 25, the percentage of contribution of each individual random model parameter to the overall output uncertainty can be estimated. The approximated sensitivity coefficients with respect to individual component uncertainty, based on Eq. 25, can be derived as

$$\frac{\partial CV(Y)}{\partial CV(X_i)} \approx \frac{a_i^2 CV(X_i)}{CV(Y)} \quad (26)$$

It should be emphasized that Eqs. 25 and 26 are only approximations of the true relationship given in Eqs. 23 and 24, respectively.

Examples

This section presents two examples to demonstrate the use of the Mellin transform in hydrologic and hydraulic uncertainty analysis and reliability analysis.

Example No. 1 (Uncertainty of Flood Travel Time)

Uncertainty analyses of hydraulic computations in channel flood routing are mainly concerned with the assessment of the uncertainty feature of the computation results. In channel flood routing, the results of primary interest are the travel time of flood water, the magnitude of peak, the corresponding water-surface profile, and the area of inundation. This example examines the uncertainty of the travel time derived from using the kinematic-wave routing model. Using Manning's formula, the travel time T of a kinematic wave in a wide rectangular channel carrying a flow of Q can be determined by (Chow et al. 1988)

$$T = \frac{3}{5}\left(\frac{nB^{2/3}}{1.49S_0^{1/2}}\right)^{3/5} Q^{-2/5}L \quad (27)$$

where B = channel width; n = Manning's roughness; and L = length of channel reach. In hydrological analyses of urban drainage design, many equations used for computing the lag time or time of concentration have a form similar to Eq. 27 (Kibler 1982; Chen and Wang 1989).

In Eq. 27, the parameters on the right-hand side of the equation are treated as random variables resulting from the spatial/temporal variabilities and measuring errors. Since the travel time T is related to n, B, S_0, Q, and L in a multiplicative manner and all the parameters with uncertainty are nonnegative, the Mellin transform is applicable. Based on Eq. 27 and using Table 2, the Mellin transform of the PDF of the travel time can be expressed as

TABLE 4. Data Used in Example No. 1

Variable (1)	Distribution (2)	L^* (3)	M^* (4)	H^* (5)
n	Triangular	0.03	0.045	0.055
B (ft)	Triangular	180	200	220
S_0 (ft/ft)	Triangular	0.00025	0.00035	0.00045
Q (cfs)	Triangular	9,000	10,000	11,000
L (mi)	Triangular	99	100	101

*L, M, and H = lower bound, mode, and upper bound, respectively, of a random variable having a triangular distribution.
Note: 1 ft = 0.305 m; 1 mi = 1.609 km; 1 cfs = 0.0283 cms.

$$E(T^{s-1}) = M_T(s) = c_T^{s-1}M_n(0.6s + 0.4)M_B(0.4s + 0.6)M_{S0}(-0.3s + 1.3)M_Q(-0.4s + 1.4)M_L(s) \quad (28a)$$

where $c_T = 0.6(1.49)^{-0.6}$. More specifically, the first four moments of the travel time about the origin are

$$E(T) = M_T(2) = c_T M_n(1.6)M_B(1.4)M_{S0}(0.7)M_Q(0.6)M_L(2) \quad (28b)$$

$$E(T^2) = M_T(3) = c_T^2 M_n(2.2)M_B(1.8)M_{S0}(0.4)M_Q(0.2)M_L(3) \quad (28c)$$

$$E(T^3) = M_T(4) = c_T^3 M_n(2.8)M_B(2.2)M_{S0}(0.1)M_Q(-0.2)M_L(4) \quad (28d)$$

$$E(T^4) = M_T(5) = c_T^4 M_n(3.4)M_B(2.6)M_{S0}(-0.2)M_Q(-0.6)M_L(5) \quad (28e)$$

Depending on the distributional properties of individual random variables on the right-hand side (refer to Table 3), the moments about the origin of the travel time can be calculated. To illustrate the computations, data shown in Table 4 are used. The values of the Mellin transforms corresponding to the appropriate argument for the different parameters, the moments about the origin, and the associated central moments are given in Table 5. The values of the statistical moments so obtained, in theory, are exact rather than approximations. However, during the computation, caution should be given to the potential numerical-rounding error when random variables with relatively small uncertainty are analyzed.

Once the basic statistical moments of the travel time are determined, one might further be interested in knowing other statistical properties of the travel time such as the confidence interval, the probability that the travel time is shorter than a certain value, and so on. To obtain such information, one has to know the PDF of the travel time. In theory, the PDF of the travel time $f_T(t)$, from the one-to-one correspondence of $f_T(t)$ and $M_T(s)$, can be derived through the inverse Mellin transform on Eq. 28. However, such an inverse transform involves integration operations in the complex variable space and is an analytically formidable task (Springer 1978). As a practical alternative, some parametric PDFs are used. Normal and log-normal distributions are among those that are frequently applied for which the first two moments are sufficient to characterize them.

Two other more complicated distributions, i.e., Fisher-Cornish asymptotic expansion and Pearson distributions, were used recently to compute the quantile of the travel time (Tung 1989). The main reason for using the Fisher-Cornish

TABLE 5. Computations of Example No. 1 in Assessing Uncertainty of Travel Time

		Order of Moment of Travel Time			
		1	2	3	4
(1)	(2)	(3)	(4)	(5)	(6)
n	s	1.6	2.2	2.8	3.4
n	$M_n(s)$	1.5183E−1	2.3170E−2	3.5535E−3	5.4761E−4
B	s	1.4	1.8	2.2	2.6
B	$M_B(s)$	8.3239E+0	6.9305E+1	5.7772E+2	4.8083E+3
S_0	s	0.7	0.4	0.1	−0.2
S_0	$M_{S0}(s)$	1.0913E+1	1.1925E+2	1.3047E+3	1.4294E+4
Q	s	0.6	0.2	−0.2	−0.6
Q	$M_Q(s)$	2.5130E−2	6.3172E−4	1.5884E−5	3.9952E−7
L	s	2.0	3.0	4.0	5.0
L	$M_L(s)$	1.0000E+2	1.0000E+4	1.0000E+6	1.0000E+8
$\mu_r' = E(T^r)$	—	1.6371E+1	2.6987E+2	4.4793E+3	7.4842E+4
μ_r	—	0.0	1.8772E+0	−1.0743E−2	8.8442E+0
Mellin	—	1.6371E+1[a]	1.3701E+0[b]	−4.1768E−2[c]	2.5098E+0[d]
MFOSM	—	1.6350E+1[a]	1.3515E+0[b]	−2.9970E−1[c]	—

[a]mean.
[b]standard deviation.
[c]skewness.
[d]kurtosis.

and Pearson distributions is that the third and the fourth moments calculated from the Mellin transform are exact values rather than approximations, or are estimated from the sample data. The main features of the two distributions are described here without giving the mathematical details.

The Fisher-Cornish asymptotic expansion approximates the quantiles of any distribution by those of the normal distribution with correction given to the presence of higher moments such as skew coefficient and kurtosis, which are not equal to those for normal random variables (Fisher and Cornish 1960; Kendall et al. 1987). Using only the first two moments, the quantiles of the Fisher-Cornish expansion reduces to those of the normal distribution.

The Pearson distribution is a four-parameter distribution. It is a very general distribution that encompasses the majority of the parametric PDFs that have been commonly used in statistical analyses. Distributions such as normal, gamma, and beta are the members of the Pearson family. The type of distribution in the Pearson system can be determined on the basis of the values of skew coefficient and kurtosis. A chart has been prepared for that purpose (Kendall et al. 1987). Parameters in the PDF can be determined by relating them to the first four moments (Kendall et al. 1987). Solomon and Stephens (1978) showed that the Pearson distribution gives an excellent approximation to the long tail of a distribution when the first four moments are known exactly.

Based on Table 5, it is observed that the travel time resulting from the data in Table 4, strictly speaking, is not normal because its skew coefficient is not zero and kurtosis is not equal to 3. In fact, the negative skewness indicates that the distribution of the travel time is not log-normal, either.

TABLE 6. The 90% and 95% Confidence Intervals of Travel Time (Days)

Distribution (1)	Confidence Intervals 90% (2)	95% (3)
Normal	(14.12, 18.63)	(13.69, 19.06)
Log-normal	(14.22, 18.72)	(13.85, 19.22)
Fisher-Cornish	(14.09, 18.62)	(13.70, 18.98)
Pearson	(14.09, 18.61)	(13.73, 18.95)

For comparison, the 90% and 95% confidence intervals of the travel time under different distributional assumptions are shown in Table 6. It is found that the confidence intervals derived from using the Fisher-Cornish expansion and Pearson distribution are practically identical. Because log-normal distribution assumes a positive skewness for the travel time, its confidence intervals shift to the right compared with the other three distributions considered. Considering the negative skewness of the travel time, the upper bound of the confidence interval resulting from Fisher-Cornish and Pearson distributions is smaller than that of normal distribution.

Comparing the bottom two rows of Table 5, the expected flood-arrival time and its standard deviation estimated by the MFOSM method are close to those estimated by the Mellin transform. However, the skew coefficient estimated by the MFOSM method is significantly larger than that calculated by the Mellin transform. For this particular example, the use of the MFOSM method with an adoption of a normal distribution, as is commonly done, yields an accurate result because the skew coefficient is very close to zero. However, the use of log-normal distribution might lead to a biased result.

Table 7 contains the sensitivity coefficients with respect to the uncertainty of individual input parameters in Eq. 27. Both Eqs. 24 and 26 indicate that the uncertainty in roughness factor has the dominant effect on the total uncertainty in estimating flood travel time. The two equations fail to agree on which variable should be ranked as the second; Eq. 24 picks the channel slope while Eq. 26 selects the top width. The magnitudes of sensitivity coefficients, except channel top width, computed by the two equations are close.

Example No. 2 (Risk Analysis of Storm Sewer Design)

Consider the design of a storm sewer system. The sewer flow carrying capacity Q_c is determined by the Manning's formula

TABLE 7. Sensitivity Coefficients of Stochastic Input Parameters in Example No. 1

Method (1)	Stochastic Input Parameters: Roughness (n) (2)	Width (B) (3)	Slope (S_0) (4)	Discharge (Q) (5)	Length (L) (6)
Mellin	0.52119	0.07751	0.13091	0.08316	0.04917
MFOSM	0.50991	0.22663	0.12544	0.07804	0.04883

$$Q_C = \frac{0.463}{n}\lambda_m d^{8/3} S_0^{1/2} \quad (29)$$

where n = Manning's roughness; λ_m = model correction factor to account for the model uncertainty; d = actual pipe diameter; and S_0 = pipe slope. The inflow Q_L to the sewer is surface runoff whose peak discharge is estimated by the rational formula

$$Q_L = \lambda_L CIA \quad (30)$$

in which λ_L = correction factor for model uncertainty; C = runoff coefficient; I = rainfall intensity; and A = runoff-contribution area.

In practice, it is reasonable to assume that all the parameters on the right-hand side of Eqs. 29 and 30 are subject to uncertainty. The sewer capacity Q_C and peak inflow Q_L from surface runoff, consequently, cannot be quantified with absolute certainty. The risk of the sewer not being able to accommodate inflow can be computed by

$$p_f = Pr[Q_C < Q_L] = Pr[Z < 0] \quad (31)$$

in which p_f = risk (probability of failure); $Pr[\ ]$ = probability operator; and $Z = Q_C - Q_L$, a performance variable.

Solving p_f requires knowledge about the statistical properties of Z. With the assumption of statistical independence of all stochastic parameters involved, the sewer capacity Q_C and peak surface runoff Q_L are uncorrelated. The statistical moments of Z about the origin, in terms of those of Q_C and Q_L, can be obtained as

$$E(Z) = E(Q_C) - E(Q_L) \quad (32)$$

$$E(Z^2) = E(Q_C^2) - 2E(Q_C)E(Q_L) + E(Q_L^2) \quad (33)$$

$$E(Z^3) = E(Q_C^3) - 3E(Q_C^2)E(Q_L) + 3E(Q_C)E(Q_L^2) - E(Q_L^3) \quad (34)$$

$$E(Z^4) = E(Q_C^4) - 4E(Q_C^3)E(Q_L) + 6E(Q_C^2)E(Q_L^2) - 4E(Q_C)E(Q_L^3) + E(Q_L^4) \quad (35)$$

As can be seen, the moments of Z are functions of moments of Q_C and Q_L, which, in turn, are functions of statistical properties of basic stochastic parameters in Eqs. 29 and 30. Note that both Q_C and Q_L are functions of several nonnegative stochastic model parameters and these functional forms are multiplicative. Hence, the moments of various orders of Q_C and Q_L on the right-hand sides of Eqs. 32–35 can be obtained by the Mellin transform. More specifically, the moments about the origin of Q_C and Q_L can, respectively, be expressed through the Mellin transform as

$$E(Q_C^k) = M_{Q_C}(s = k + 1)$$

$$= 0.463^{s-1} M_{\lambda m}(s) M_n(2 - s) M_d\left(\frac{8s}{3} - \frac{5}{3}\right) M_{S0}\left(\frac{s}{2} - \frac{1}{2}\right) \quad (36)$$

$$E(Q_L^k) = M_{QL}(s = k + 1) = M_{\lambda L}(s) M_C(s) M_I(s) M_A(s) \quad (37)$$

in which $M_X(w)$ represents the Mellin transform of the PDF of random variable X with an argument w. Substituting Eqs. 36 and 37 into Eqs. 32–35

TABLE 8. Data for Example No. 2

Variable (1)	Mean (2)	Standard deviation (3)	Distribution (4)
λ_m	1.100	0.098	Triangular
n	0.015	0.00083	Gamma
d (ft)	3.000	0.123	Triangular
S_0 (ft/ft)	0.005	0.00082	Triangular
λ_L	1.000	0.123	Triangular
C	0.825	0.051	Triangular
I (in./hr)	4.000	0.612	Triangular
A (acre)	10.00	0.408	Triangular

Note: 1 ft = 0.305 m; 1 in. = 2.54 cm; 1 acre = 4,047 m^2.

using an appropriate value for the argument s, one can compute the moments about the origin of performance variable Z. Once $E(Z^k)$ are computed, the mean, variance, skew coefficient, and kurtosis of Z can be obtained by using Eqs. 7–11.

To estimate the probability of failure p_f, one has to know the PDF of performance variable Z. In general, the exact PDF of Z cannot be easily derived. Hence, a simpler probability distribution is assumed for Z in practice. If a normal or log-normal distribution is used, then only the first two moments are needed. To use a more complicated PDF such as the Pearson distribution or the Edgeworth asymptotic expansion (Abramowitz and Stegun 1972; Kendall et al. 1987), the required information about higher-order moments can be obtained with few additional computations. Similar to the Fisher-Cornish expansion, the Edgeworth expansion approximates the probability and PDF of a nonnormal random variable by the normal variable with correction terms for higher-order moments. The difference between the two methods is that the Fisher-Cornish expansion approximates the quantiles, while the Edgeworth expansion approximates the value of probability or PDF.

TABLE 9. Moments of Q_c, Q_L, and Z in Example No. 2

Order of moments (1)	Variables Q_c (2)	Q_L (3)	Z (4)
1[a]	4.505E+1	3.300E+1	1.205E+1
2[a]	2.090E+3	1.137E+3	2.543E+2
3[a]	9.980E+4	4.086E+4	5.756E+3
4[a]	4.902E+6	1.526E+6	1.550E+5
1[b]	45.05	33.00	12.05
2[b]	60.80	48.46	109.30
3[b]	1.846E+2	1.247E+2	59.95
4[b]	1.135E+4	6.919E+3	3.595E+4
Skew	0.389	0.370	0.052
Kurtosis	3.070	2.946	3.011

[a]Moments about the origin.
[b]Central moments.

To illustrate the application of the method, numerical data presented in Table 8 are used. Table 9 shows the computational results for the statistical moments of sewer capacity Q_C, runoff peak discharge Q_L, and performance variable Z. The probabilities of failure computed by using normal, log-normal, Edgeworth expansion, and Pearson distribution are 0.00275, 0.00192, 0.00213, 0.00299, respectively. The maximum difference in risk values among these four distributions is about 36%. Further numerical investigations by changing the pipe size indicate that the percent difference in risk values decreases as the risk value increases.

CONCLUSIONS

This paper introduces a mathematical transform technique called the Mellin transform, which is potentially useful in uncertainty analyses of hydrologic and hydraulic problems. Many equations used in engineering designs in hydrology and hydraulics are empirically derived involving parameters subject to uncertainty. The Mellin transform is especially attractive and simple to use when the dependent random variable is related to several independent random variables in a multiplicative manner. Under such circumstances, exact values of the moments of any order can be derived with simple algebraic manipulations. In fact, many equations in hydrologic and hydraulic computations are of this nature, to which the Mellin transform is applicable.

The description of the mathematics of the Mellin transform in the paper assumes that random model parameters are nonnegative and uncorrelated. In actuality, such restrictions for the random variables are not absolutely required. However, the removal of such restrictions increases mathematical manipulation tremendously (Springer 1979). Using integral transform techniques requires specifying the PDFs of stochastic input parameters, not just the first two statistical moments. Incorporation of correlations among input parameters requires that the joint PDFs are specified. This could be difficult in practice.

In uncertainty analysis, the objective often focuses on quantifying the statistical moments of model output when stochastic input parameters are involved. The statistical moments of model output are dependent on the moments as well as the PDFs of model parameters. Given the values of the first two moments of a random variable, there could be several PDFs yielding the same moments. Methods like the MFOSM technique then are unable to examine the impact of the form of PDF on the moments of the output. To investigate the effect of PDFs of input parameters on the statistical moments of the model output, the Mellin transform and other integral transform techniques, when appropriate, can provide an analytically easy tool to do so.

Finally, the Mellin transform yields an expression of uncertainty in model output that is amenable for analytical sensitivity analysis. Results of sensitivity analysis provide useful information in directing future data-collection efforts in an attempt to reduce uncertainty in model output.

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Appendix I. References

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Appendix II. Notation

The following symbols are used in this paper:

A = drainage area (acres);

a_i = constant associated with the ith stochastic input parameter;
B = top width of channel (ft);
C = runoff coefficient;
$CV(\ )$ = coefficient of variation;
d = sewer pipe diameter (in.);
$E(\ )$ = expectation operator;
I = rainfall intensity (in./hr);
L = length of channel reach;
$M_x(\ )$ = Mellin transform of function $f(x)$;
n = Manning's roughness coefficient;
$Pr[\ ]$ = probability operator;
p_f = probability of failure;
Q = discharge (cfs);
Q_c = sewer capacity (cfs);
Q_L = surface runoff loading to sewer system (cfs);
S_0 = channel slope or pipe slope;
T = travel time of flood (days);
$\mathrm{Var}(\ )$ = variance operator;
X_i = the ith stochastic model input parameter;
Y = model output (random variable);
γ = skew coefficient;
κ = kurtosis;
λ_L = model uncertainty term for runoff loading equation;
λ_m = model uncertainty term for sewer-capacity equation;
μ = mean value;
μ_r = the rth moment of random variable about any arbitrary reference point x_0;
μ_r = the rth central moment;
μ'_r = the rth moment about the origin;
σ = standard deviation; and
$*$ = convolution operator.