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Proceedings

1989 WWRC-89-47

In

Finite Element Analysis in Fluids: Proceedings of the Seventh International Conference on Finite Element Methods in Flow Problems

Submitted by

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FINITE ELEMENT ANALYSIS IN FLUIDS

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Proceedings of the Seventh International Conference on Finite Element Methods in Flow Problems

APRIL 3 – 7, 1989

The University of Alabama in Huntsville Huntsville, Alabama

T. J. Chung, and Gerald R. Karr, Editors

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PARALLEL COMPUTING SPEEDUPS FOR ALTERNATING DIRECTION COLLOCATION

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ABSTRACT

We apply finite-element collocation to the two-dimensional advection-diffusion equation. Collocation offers savings over other finite-element techniques in that matrix elements are found by point evaluations rather than integrations. Additional computer time and storage is saved by application of an alternating direction process, which allows a multidimensional problem to be solved as a sequence of one-dimensional problems. Since these one-dimensional problems are independent, the speed of the method is enhanced further through use of a parallel computing architecture.

1. INTRODUCTION

Alternating direction (AD) methods have been formulated for the numerical solution of partial differential equations since their introduction in 1955 by Peaceman and Rachford [1]. In 1970 Douglas and Dupont [2] developed the alternating direction Galerkin method. More recently, the alternating direction collocation (ADC) method has appeared in several formulations by Bangia et al. [3], Chang and Finlayson [4], Hayes [5], Celia et al. [6], Celia [7], and Celia and Pinder [8].

We examine Celia's ADC for the two-dimensional advection-diffusion equation. Of special interest here is the amenability of the procedure to implementation on parallelarchitecture computers. The paper has the following structure: Section 2 briefly reviews finite-element collocation using a bicubic Hermite basis; Section 3 discusses the AD method applied to collocation; Section 4 concludes the paper with an examination of the method's performance on a parallel computer.

2. REVIEW OF FINITE-ELEMENT COLLOCATION

Consider the following problem posed on the spatial domain $\Omega = (a, b) \times (c, d)$:

(a) $\partial_t u + \mathbf{v} \cdot \nabla u - \nabla \cdot (D\nabla u) = 0$, $(x, y, t) \in \Omega \times (0, \infty)$,

(1)

- (b) $u(x, y, 0) = u_I(x, y), (x, y) \in \Omega,$
- (c) $u(x,y,t) = u_B(x,y,t), (x,y) \in \partial\Omega, t \geq 0.$

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Here $\mathbf{v} = \mathbf{v}(x, y)$ represents fluid velocity; D = D(x, y) is a diffusion coefficient, and $\mathbf{u} = \mathbf{u}(x, y, t)$ represents solute concentration. We apply finite-element collocation to the following semidiscrete analog:

$$u^{n+1} - u^n + k[\mathbf{v} \cdot \nabla u^{n+\ell} - \nabla \cdot (D\nabla u^{n+\ell})] = 0, \qquad (2)$$

where integer superscripts indicate time level, $(\cdot)^{n+\theta} \equiv \theta(\cdot)^{n+1} + (1-\theta)(\cdot)^n$, $0 \le \theta \le 1$, and k signifies the time step.

We begin by establishing a grid on Ω . Let $\Delta_x = \{x_i = a + ih_x, i = 0, \dots, N_x\}$ and $\Delta_y = \{y_j = c + jh_y, j = 0, \dots, N_y\}$, where $h_x = (b - a)/N_x$ and $h_y = (c - d)/N_y$. The Hermite piecewise cubics on these grids are

$$\mathcal{M}_1^3(\Delta_x) = \{f \in C^1(\bar{\Omega}) \mid f|_{[x_{i-1},x_i]} \text{ is cubic, } i = 1,\ldots,N_x\},\$$

and similarly for $M_1^3(\Delta_y)$. As Prenter [9] shows, each of these spaces has an interpolating basis $\{h_{0i}, h_{1i}\}_{i=0}^{N_0 \text{ or } N_y}$, every element of which has support confined to at most two adjacent subintervals $[x_{i-1}, x_i]$ or $[y_{j-1}, y_j]$.

At each time level n we compute an approximate solution $\hat{u}^n(x, y)$ belonging to the tensor-product trial space

$$\mathcal{M} = \{ v \in \mathcal{M}_1^3(\Delta_x) \otimes \mathcal{M}_1^3(\Delta_y) \mid v(x,y) = u_B(x,y) \text{ for } (x,y) \in \partial \Omega \}.$$

Each function in M obeys the boundary conditions (1c) and has the form

$$\hat{u}^{n}(x,y) = \sum_{i=0}^{N_{x}} \sum_{j=0}^{N_{y}} \left[\hat{u}^{n}(x_{i},y_{j}) H_{00ij}(x,y) + \frac{\partial \hat{u}^{n}}{\partial x}(x_{i},y_{j}) H_{10ij}(x,y) \right. \\ \left. + \frac{\partial \hat{u}^{n}}{\partial y}(x_{i},y_{j}) H_{01ij}(x,y) + \frac{\partial^{2} \hat{u}^{n}}{\partial x \partial y}(x_{i},y_{j}) H_{11ij}(x,y) \right],$$

where $H_{\ell m i j}(x, y) = h_{\ell i}(x) h_{m j}(y)$. At t = 0 we form \hat{u}^0 by projecting the initial function u_l onto \mathcal{M} . These criteria specify \hat{u}^0 completely and determine $4(N_x + N_y + 1)$ of the $4(N_x + 1)(N_y + 1)$ nodal coefficients for $\hat{u}^1, \hat{u}^2, \ldots$.

To determine the remaining $4N_xN_y$ degrees of freedom at each time level n + 1, we first form the residual

$$R^{n+1} = \hat{u}^{n+1} - \hat{u}^n + k \left[\mathbf{v} \cdot \nabla \hat{u}^{n+\theta} - \nabla \cdot (D \nabla \hat{u}^{n+\theta}) \right].$$

We then pick a collection $\{(\bar{x}_1, \bar{y}_1), (\bar{x}_1, \bar{y}_2), \dots, (\bar{x}_{2N_s}, \bar{y}_{2N_s})\}$ of collocation points and force $R^{n+1}(\bar{x}_k, \bar{y}_\ell) = 0$ at each. To obtain optimal $O((h_x + h_y)^4)$ error estimates, we choose \bar{x}_k and \bar{y}_ℓ to be the two-point Gauss-quadrature abscissae on each subinterval $[x_{i-1}, x_i]$ or $[y_{j-1}, y_j]$.

3. THE ALTERNATING DIRECTION METHOD

To obtain a matrix that can be factored into AD form, we first perturb Equation (2) by a term that is $O(k^2)$ to get

$$u^{n+1} - u^n + k(\mathcal{L}_z + \mathcal{L}_y)u^{n+\theta} + k^2\theta^2(\mathcal{L}_z\mathcal{L}_y)(u^{n+1} - u^n) = 0, \qquad (3)$$

where

$$\mathcal{L}_{z} = v_{z}\partial_{z} - \partial_{z}(D\partial_{z})$$
 and $\mathcal{L}_{y} = v_{y}\partial_{y} - \partial_{y}(D\partial_{y})$.

Rearranging Equation (3) and factoring gives

$$(1+k\theta\mathcal{L}_y)(1+k\theta\mathcal{L}_z)(u^{n+1}-u^n)=-k(\mathcal{L}_z+\mathcal{L}_y)u^n.$$

Now we can solve $(1 + k\theta \mathcal{L}_y)z = -k(\mathcal{L}_x + \mathcal{L}_y)u^n$, followed by $(1 + k\theta \mathcal{L}_z)(u^{n+1} - u^n) = z$.

When we substitute Hermite bicubic trial functions for \hat{u} , we get a matrix equation $K\mathbf{u}^{n+1} = \mathbf{r}^n$, where \mathbf{u}^{n+1} is the vector of time increments for the unknown nodal coefficients of \hat{u} . Consider a typical entry of K:

$$\left\{ \left[1 + k\theta(\mathcal{L}_{z} + \mathcal{L}_{y}) + k^{2}\theta^{2}(\mathcal{L}_{z}\mathcal{L}_{y}) \right] H_{\sigma} \right\} (\bar{x}_{k}, \bar{y}_{\ell}), \tag{4}$$

where H_{σ} is shorthand for some basis function H_{tmij} . Each $H_{\sigma}(x, y) = h_{\sigma}(x)h_{\beta}(y)$, with $\alpha = (i, r)$ and $\beta = (j, s)$, so we can expand the expression (4) and factor it to get

$$[h_{\alpha}(\bar{x}_{k}) + k\theta(\mathcal{L}_{z}h_{\alpha})(\bar{x}_{k})] \cdot [h_{\beta}(\bar{y}_{\ell}) + k\theta(\mathcal{L}_{y}h_{\beta})(\bar{y}_{\ell})].$$

If we number the nodes along the lines $x = \bar{x}_k$, we can use this observation to factor the $4N_xN_y \times 4N_xN_y$ matrix K as follows:

$$K = Y \cdot X = \begin{bmatrix} Y_{1,1} & & \\ & \ddots & \\ & & Y_{2N_{*},2N_{*}} \end{bmatrix} \cdot \begin{bmatrix} X_{1,1} & \cdots & X_{1,2N_{*}} \\ \vdots & & \vdots \\ X_{2N_{*},1} & \cdots & X_{2N_{*},2N_{*}} \end{bmatrix}.$$

Each $2N_{\psi} \times 2N_{\psi}$ block $Y_{j,j}$ has the five-band structure of a one-dimensional collocation matrix, and its entries depend only on the y-coordinates of collocation points. We can solve the matrix equation $Ku^{n+1} = r^n$ by the following procedure.

- 1. Order the nodes vertically and solve $Yz = r^n$ by solving the independent problems $Y_{j,j}z_j = r_j^n, j = 1, ..., 2N_z$.
- 2. Reorder the nodes horizontally to convert z to z^* . This operation transforms X to a block-diagonal form X^* whose blocks $X_{i,i}^*$ have one-dimensional structure.

3. Solve $X^* u^{n+1} = z^*$ by solving the independent systems $X_{i,i}^* u_i^{n+1} = z_{i,i}^* = 1, \dots, 2N_v$.

Each of the "one-dimensional" systems in steps 1 and 3 is independent of any other. Therefore these steps can be done concurrently.

4. IMPLEMENTATION ON A PARALLEL COMPUTER

We have implemented ADC on an Alliant FX/8 parallel processing computer. The Alliant is an eight-processor, shared-memory machine with optimization capability for both concurrent and vector programming. The machine allows users to control concurrency within a Fortran code through the use of compiler directives. The following is a description of the code outlined in Steps 1-3 of Section 3. The compiler directives themselves begin with the flag CVD\$ starting in the first column of code. 950

```
Initialize &", set n = 0
        Begin time level n + 1
          CNCALL (Compiler directive to permit the concurrent execution of the following loop
CVDU.
                      containing a reference to an external procedure.)
           DO for each j = 1, \ldots, 2N_s
              CALL YSWEEP (Constructs the system Y_{j,j}\mathbf{z}_j = \mathbf{r}_j^n, solves it and saves the results.)
           END DO
           CALL RENUM (Reorders s to get s')
CVD$L
          CNCALL
           DO for each i = 1, \ldots, 2N_{y}
              CALL XSWEEP (constructs the system X_{i,i}^* u_i^{n+1} = z_i^*, solves it and updates the
                               appropriate coefficients of 0 to time level n + 1.)
           END DO
        End time step
CVD$R NOCONCUR (Directive to supress concurrency until the end of the routine.)
        SUBROUTINE YSWEEP
CVD$R NOCONCUR
        SUBROUTINE XSWEEP
```

One measure of how well the algorithm makes use of the machine's parallel capabilities is the speedup. Speedup for n processors is the ratio of the time needed by one processor to the time used by n processors to perform a set of tasks in parallel. For a perfectly parallel algorithm requiring no overhead to monitor and schedule the various processors, the speedup for n processors would be n. Figure 1 shows the speedup curve for this algorithm, excluding initialization. The speedup for eight processors is 7.27. Clearly, ADC makes very good use of the shared-memory parallel architecture.

To confirm that ADC gives useful approximations, Figures 2 and 3 show solution plots for two different problems. Figure 2 shows the results of a rotating plume problem on $\Omega = (-1, 1) \times (-1, 1)$, with $N_x = N_y = 40$ and k = 0.004. Here, $\mathbf{v} = 2\pi(-y, x)$ is a circular velocity field, D = 0, and the initial concentration plume $u_1(x, y)$ is a "Gauss hill" with center at (0, -0.6) and standard deviation $\sigma = 0.066$. Figure 3 displays the results of an advection-diffusion problem on $\Omega = (0, 1) \times (0, 1)$, with $N_x = N_y = 20$ and k = 0.004. The diffusion coefficient is D = 0.00385, and $\mathbf{v}(x, y) = 2e^{ry}(-y, x)$. Here, u_f is a "Gauss hill" with $\sigma = 0.05$ centered at (0.75, 0.25). In both problems the global error is less than $.02||u||_{\infty}$.

ACKNOWLEDGMENTS

The Wyoming Water Research Center supported this work. We also received support from NSF grant RII-8610680 and ONR contract 0014-88-K-0370.

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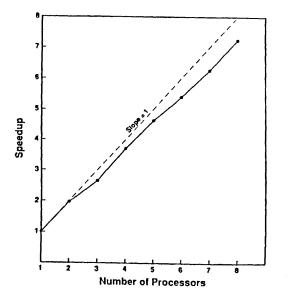


Figure 1. Speedup curve for ADC using the Alliant FX/8 shared-memory architecture.

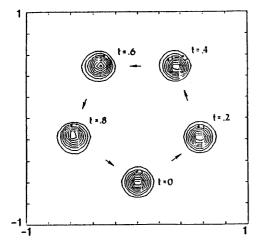
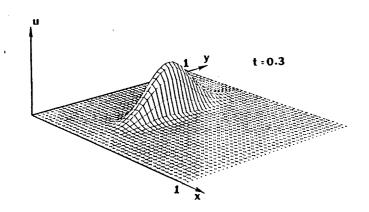


Figure 2. Concentration contours for the purely advective rotating plume problem at various time levels. Contour interval is 0.1.



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Figure 3. Plot of concentration distribution at t = 0.3 for an advection-diffusion problem with potential flow.

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