GROUNDWATER MANAGEMENT BY CHANCE-CONSTRAINED MODEL

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To convert	to	Divide by
cubic meters (m^3)	acre-ft (acre-ft)	1.23×10^3
hectares (ha)	acres (acres)	0.405
kilograms (kg)	pounds mass (lbm)	0.454
kilometers (km)	miles (miles)	1.61
kilopascals (kPa)	pounds force per square inch (psi)	6.89
liters (L)	U.S. gallons (gal)	3.79
millimeters (mm)	inches (in.)	25.4
newtons (N)	kilograms force (kgf)	9.81
newtons (N)	pounds force (lbF)	4.45

Groundwater Management by Chance-Constrained Model

By Yeou-Koung Tung,[1] A. M. ASCE

Abstract: A stochastic groundwater management model for a confined, homogenous, and nonuniform aquifer is developed using the concept of response function in the linear system theory. The Cooper-Jacob equation is used to develop the unit response function. The model explicitly considers the random nature of transmissivity and storage coefficient, which enables the determination of optimal pumping pattern in a well field subject to a specified system performance reliability requirement. A hypothetical example is utilized to demonstrate applicability of the model. Model results affected by reliability requirement and uncertainty level of aquifer parameters were examined. A post-optimality simulation is conducted to examine the performance of the model and to further assess its usefulness.

Introduction

Highly variable surface water in arid and semi-arid regions has placed groundwater in a major role in most water supply systems. However, due to lack of proper management, many groundwater aquifers were depleted and contaminated. Benefiting from the advancement of geophysical knowledge on subsurface flow phenomena and computer capability, effective management of groundwater aquifer of various complexities has become practical and viable.

Literature on optimal groundwater management can be found elsewhere. Basically, the methodology can be classified into simulation and direct optimization. Groundwater managements using simulation approach (36) generally employ numerical groundwater models based upon either finite difference (28,32) or finite element (27) schemes. Because groundwater simulation models mainly describe the stress-response relationship of an aquifer system, the use of the simulation approach to seek optimal management scheme requires trial and error, which could be very time consuming and laborious.

Direct optimization approach, on the other hand, includes some types of automatic optimal seeking algorithms. Depending on how detailed the system is to be modeled, groundwater management models using the direct optimization approach can be categorized into lumped-parameter and distributed-parameter models. Lumped-parameter models are mainly concerned with the temporal allocation of water, which generally is computationally simpler. Examples of lumped groundwater management models can be found elsewhere (6,7,10). If management decisions concern both temporal and spatial aspects of water allocations and system behavior in groundwater aquifers, a distributed-parameter model should be employed. Optimal groundwater management models with

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distributed-parameter capabilities can be approached either by an embedding technique or response-matrix technique (18,31). The embedding technique directly incorporates the groundwater flow equation as constraints in an optimization framework (1–3), while the response-matrix technique utilizes the concept of influence function and linear system theory (8,16,17,19,22,26,34). A review of the two techniques in groundwater management was recently given by Gorelick (18).

Like any other resource management, groundwater management is generally done in the environment where uncertainties exist. Uncertainty in groundwater management may be ascribed mainly to lack of perfect knowledge about an aquifer system, inherent variability of system parameters and flow characteristics (12,15), and other factors such as costs and revenues of the project, engineering design, and operation of the system. As a result, the existence of uncertainties limits our capability to predict system behavior with definiteness under various management decisions. Several studies were made to consider the effect of stochasticity in groundwater management. Burt (7) incorporated random recharge or stream in his economic study. Maddock (23) considered the effect of random demand in a distributed-parameter aquifer model. Recently, Flores et al. (11) developed a physical-based lumped stochastic model for managing a stream-aquifer system.

In groundwater management, the selection of an appropriate model for analyzing cause-and-effect relationships of subsurface water flow is largely dependent on the budgetary condition and data availability of the groundwater system. Capabilities of a sophisticated distributed-parameter groundwater model is well-understood. However, meaningful results can be generated only if there are sufficient amounts of data of good quality available. Bredehoef and Young (5) stated that, "The limited resources available to the project precluded any detailed field studies of hydrologic, legal, and economic relationships necessary to represent a specific area accurately." Bathala et al. (4) investigated the problems encountered in the formulation of digital simulation models, particularly those related to the data, manpower, and computational expenditure. They concluded that the results from digital simulation models developed by using limited available data should be interpreted with caution. In addition to the inherent random process of subsurface flow, data on most groundwater basins are lacking. This is particularly true for an undeveloped basin during earlier stages of planning. In such circumstances, there are only a few pumping tests, boundary conditions regarding quantities and locations of recharges and discharges are difficult to estimate, and available information often contains errors in observations and interpretations that introduce additional uncertainties. As a result, the use of a realistic distributed-parameter aquifer model may not be necessary. Prickett (29) pointed out that "In any event, choosing an overly sophisticated model which doesn't fit the problem is a case of applying the wrong model." Recently, he (30) addressed the need in developing a large group of models aimed at solving problems in the range of simple to moderate complexity.

Furthermore, there have been some studies made showing the evidence that the use of a simplified subsurface flow model in groundwater management might be adequate. Young and Bredehoeft (36) observed

in their simulation study that management decisions were relatively insensitive to the change in hydraulic conductivity of the aquifer. Maddock (23) also performed a sensitivity analysis on a distributed-parameter model in his groundwater management study and found that results were most sensitive to economic factors rather than to aquifer parameters. Recently, McElwee and Yukler (24) analyzed sensitivity of groundwater models of various complexities with respect to variations in transmissivity and the storage coefficient. They observed that about 20% change in transmissivity and storage coefficient would only result in 5% change in drawdown. Also, the results of sensitivity analysis on a two-dimensional distributed-parameter groundwater flow model in a homogenous aquifer are basically the same as those obtained from a simple Theis equation.

Due to lack of data, and relative insensitivity of management decisions to variation of aquifer properties, it seems that the use of a simple but representative groundwater model could be adequate in management problems. Furthermore, computational simplicity is an advantage of using a simple model to provide decision makers with quick but relevant solutions to management problems.

In light of the inherent randomness of subsurface flow and the existence of uncertainties in aquifer parameters, the groundwater flow systems should be treated as stochastic processes and aquifer properties be considered as random variables. To carry this argument even further, groundwater management models should, if possible, have the feature to take the random nature of the subsurface flow system into account and derive management solutions accordingly.

In this paper, the development of a simple multiple-period (transient) stochastic groundwater management model is illustrated utilizing the Cooper-Jacob equation. The model considers explicitly the random characteristics of transmissivity and the storage coefficient in a confined homogenous aquifer. The stochastic management model is formulated by transforming model constraints containing random aquifer properties to the so-called chance-constrained expression (21), which specify the reliability requirements of system performance.

Unit Response Functions

In groundwater management models constraints describing relationships between system responses and management decisions are generally included for purposes of control. The constraint equations of this type in groundwater management models presented herein utilize the approach of the unit response functions. Unit response functions describe relationships between state variables of an aquifer system such as drawdown and management decision variables such as pumpage.

The continuous form of convolution relations between aquifer drawdown and discharge for a linear flow system can be expressed as (22)

$$s(X_j, t) = \sum_{i=1}^{M} \int_0^t \beta(X_i, X_j, t - \tau) Q(X_i, \tau) d\tau \quad (1)$$

where $s(X_j, t)$ = drawdown at control point X_j at time t; $\beta(X_i, X_j, t - \tau)$

= drawdown response at control point X_j resulting from a unit impulse of pumping at point X_i during time τ; $Q(X_i, \tau)$ = pumpage of discharge well at X_i during time τ; and M = total numbers of pumping wells under consideration. The time-dependent drawdown response function, $\beta(X_i, X_j, t)$, represents incremental drawdown of each control point at X_j at time $t = \tau$ resulting from a unit impulse of pumping at each discharging well applied at time $t = 0$. When the time scale is discretized, Eq. 1 can be expressed in an equivalent form as

$$s(j,n) = \sum_{i=1}^{M} \sum_{k=1}^{n} \beta(i,j,k) Q(i, n-k+1) \quad (2)$$

where $s(j,n)$ = drawdown at the jth control point at the end of the nth period; $\beta(i,j,k)$ = response function for the kth period relating drawdown at the jth control point to unit pumpage at the ith discharging well; and $Q(i, n-k+1)$ = pumpage at the ith discharging well during the kth period, $k \le n$.

In groundwater management practices, the entire planning horizon is generally divided into operational intervals. An operation policy or management decision may vary from one operational interval to another but it generally remains the same within each operational interval. As a result, discrete formulation of convolution relation, Eq. 2, is more practical than the continuous formulation in groundwater management.

The unit response function, β, can be obtained from a distributed-parameter groundwater simulation model. Procedures of obtaining unit response functions using a simulation model were described by Heidari (19). However, when hydrogeologic information of an aquifer system is lacking or unavailable, some closed form of analytical solution to an idealized condition can be utilized to derive the unit response function. In this paper, a stochastic groundwater management model is developed for a confine, homogenous, and nonuniform aquifer with the following assumptions: (1) Aquifer is nonleaky and infinite in horizontal extent; (2) radial flow pattern; (3) wells fully penetrate the entire thickness of aquifer; and (4) piezometric head prior to pumping is uniform throughout the entire aquifer. Under the preceding assumptions, the unit response function can be obtained from the well functions (4,22):

$$\beta(i,j,k) = \begin{cases} \psi(i,j,k); & \text{for } k = 1 \\ \psi(i,j,k) - \psi(i,j,k-1); & \text{for } k \ge 2 \end{cases} \quad (3)$$

where $\psi(i,j,k) = (1/4\pi T)\, W[u(i,j,k)]$ in which $W[\ ]$ = well function and $u(i,j,k) = r_{ij}^2 S/4\pi T t_k$; r_{ij} = distance between the ith pump well and the jth control point; S = storage coefficient; T = aquifer transmissivity; and t_k = time instant at the end of the kth period.

The well function for the Theis equation can be written as

$$W[u(i,j,k)] = \int_{u(i,j,k)}^{\infty} \frac{e^{-v}}{v}\, dv \quad (4)$$

while for the Cooper-Jacob equation

$$W[u(i,j,k)] = \ln\left(\frac{2.25 T t_k}{r_{ij}^2 S}\right) \quad (5)$$

In this paper, the Cooper-Jacobs equation is utilized to demonstrate the development of a stochastic groundwater management model. A stochastic management model based on the Theis equation was also developed by the writer but will not be presented.

Deterministic Management Model

Consider the quantity aspect of groundwater management emphasizing on hydraulic response control and water supply capability of a groundwater basin. The problem is to determine the optimal pumpage and pumping pattern over a specified planning horizon such that undesirable consequences do not occur. In general, undesirable consequences such as depletion of aquifer and land subsidence can be avoided by properly controlling aquifer drawdown.

Since the response function characterizes an aquifer pumpage drawdown relationship, a groundwater management model can be very easily formulated once the response functions are defined. Without considering the random nature of aquifer properties, the deterministic management model can be stated as follows:

$$\text{Maximize} \quad \sum_{i=1}^{M} \sum_{n=1}^{N} Q(i,n) \quad (6)$$

$$\text{Subject to} \quad \sum_{i=1}^{M} \sum_{k=1}^{n} \beta(i,j,k) Q(i, n-k+1) \le s^*(j,n); \quad \text{for all } j \text{ and } n \quad (7)$$

$$\sum_{i=1}^{M} Q(i,n) \ge D(n); \quad \text{for all } n \quad (8)$$

in which $D(n)$ = water demand during the nth period. When the objective function like Eq. 6 is used, the model tends to withdraw water as much as it can be allowed by the drawdown constraints, i.e. Eq. 7. The foregoing management model follows the well-known linear programming (LP) format, which can be solved very easily by the simplex algorithm. The problem size depends on the number of time periods, pumping wells, and control points.

Probabilistic Consideration of Model Elements with Uncertainty

Values for transmissivity and storage coefficient are derived from a pump well test, and as such a test provides in situ values of aquifer parameters averaged over a large and representative aquifer volume (13), T and S should be treated as random variables. Consequently, the response function β and the left-hand side of the drawdown constraint are random in nature because they contain random variables of T and S. This implies that the compliance of constraints at each control point cannot be assured with certainty. Thus, it is more appropriate and realistic to examine the constraint performance probabilistically. In a stochastic environment, it is operationally feasible to specify limitations on allowable risk or required reliability of constraint performance. Now, if we impose a restriction on that drawdown at any control point j at the end of the nth period resulting from pumping operation over the entire well field cannot exceed a specified value $s^*(j,n)$ with a reliability $\alpha(j,n)$, the drawdown constraints then can be expressed as

$$Pr\left\{\sum_{i=1}^{M}\sum_{k=1}^{n}\beta(i,j,k)Q(i,n-k+1)\le s^*(j,n)\right\}\ge\alpha(j,n);\quad\text{for all } j \text{ and } n \qquad (9)$$

A probabilistic statement of drawdown constraint like Eq. 9 is not mathematically operational and further modification or transformation is required. To make Eq. 9 mathematically operational, it is necessary to assess statistical properties of random terms in chance-constrained equations.

First-Order Analysis.—There have been a number of field investigation and laboratory experiments, cited in Ref. 12, assessing the probability distributions of aquifer transmissivity and hydraulic conductivity. Most findings indicate that hydraulic conductivity has a log-normal distribution. Because the response function, β, is a nonlinear function of transmissivity and storage coefficient, the probability density function of β as well as drawdown at each control point cannot be easily assessed. Therefore, it is decided that the first-order analysis is applied to estimate statistical properties of the response function and drawdown at each control point.

First-order analysis is a useful method to estimate statistical characteristics such as the mean and variance of a function involving random components. The method has been applied in many aspects of water resources problems (9,33). In first-order analysis, the function containing random variables is expanded in Taylor series about the mean values of random variables, i.e.

$$f(\underset{\sim}{y})=f(\mu)+\sum_{p=1}^{P}\frac{\partial f(y)}{\partial y_p}\bigg|_{y=\mu}(y_p-\mu_p)+\epsilon \qquad (10)$$

in which $f(\underset{\sim}{y})$ = a function involving P random variables; μ = a vector of mean values of P random variables; and ϵ = higher order terms in Taylor expansion. Neglecting the higher order terms in Eq. 10 and assuming independence of random variables involved, the mean and variance of the function $f(\underset{\sim}{y})$ can be approximated as

$$E[f(\underset{\sim}{y})]\approx f(\mu) \qquad (11)$$

$$\text{var}\,[f(\underset{\sim}{y})]\approx\sum_{p=1}^{P}\left[\frac{\partial f(\underset{\sim}{y})}{\partial x_p}\bigg|\,\underset{\sim}{y}=\mu\right]^2\sigma_p^2 \qquad (12)$$

in which $E[\ ]$ and var $[\ ]$ are the expectation and variance, respectively; and σ_p^2 is the variance of the pth random variable.

Derivations of statistical properties of drawdown at each control point assuming independency of transmissivity and the storage coefficient are given in Appendix I and the results are given as follows:

$$E[s(j,n)]\approx\sum_{i=1}^{M}\sum_{k=1}^{n}\bar\beta(i,j,k)Q(i,n-k+1) \qquad (13)$$

$$\text{var}\,[s(j,n)]\approx\left[\sum_{i=1}^{M}\sum_{k=1}^{n}\bar A(i,j,k)Q(i,n-k+1)\sigma_T\right]^2+\left[\sum_{i=1}^{M}\sum_{k=1}^{n}\bar B(i,j,k)Q(i,n)\sigma_S\right]^2 \qquad (14)$$

in which $E[s(j,n)]$ and var $[s(j,n)]$ = respectively, the mean and variance of drawdown at control point j at the end of the nth period; σ_T^2 = the variance of the transmissivity; σ_S^2 = the variance of the storage coefficient; $\bar\beta(i,j,k)$, $\bar A(i,j,k)$, and $\bar B(i,j,k)$ = coefficients that are function of the mean transmissivity and storage coefficient as shown in Appendix I. It is shown in Eqs. 13 and 14, the mean of drawdown is a linear function of the pumpage while the variance of drawdown is a quadratic function of the pumpage. Derivation of Eqs. 13 and 14 enables the development of deterministic equivalent of Eq. 9, as shown in the next section, which is mathematically operational and the random characteristics of the aquifer properties are explicitly incorporated in the management model.

Stochastic Management Model

Since the total drawdown at any control point is the sum of the drawdown created by many individual pump wells, the total drawdown at each control point can be assumed to have a normal distribution (loose use of the central limit theorem, CLT) with mean and variance given by Eqs. 13 and 14, respectively. Under the normality assumption the original chance-constrained equation (9) can be expressed as

$$Pr\left\{Z\le\frac{s^*(j,n)-E[s(j,n)]}{\sqrt{\text{var}\,[s(j,n)]}}\right\}\ge\alpha(j,n) \qquad (15)$$

where Z = a standard normal random variate with mean zero and unit variance. By substituting Eq. 13 into 15, then an equivalent expression can be written as

$$\sqrt{\text{var}\,[s(j,n)]}\,F^{-1}[\alpha(j,n)]+\sum_{i=1}^{M}\sum_{k=1}^{n}\bar\beta(i,j,k)Q(i,n-k+1)$$

$$\le s^*(j,n);\quad\text{for all } j \text{ and } n \qquad (16)$$

in which $F^{-1}[\alpha(j,n)]$ = a standard normal deviate corresponding to the normal cumulative distribution function of $\alpha(j,n)$.

Note that the first term in Eq. 16 involves a square root of the variance of drawdown at each control point which, in turn, is a quadratic function of unknown decision variables Q's (see Eq. 14). The deterministic equivalent of a chance-constrained equation is nonlinear and the use of the LP technique for problem solving is prohibited. However, a linearization procedure called quasi-linearization can be employed to linearize the nonlinear terms in Eq. 16. The linearization procedure is similar to the one used by Willis (35).

In the process of linearization, the nonlinear terms in Eq. 16 is expanded in Taylor series about any arbitrary pumping rate, say $[Q^0(i,n$

$- k + 1]$ for all $i = 1, \ldots, M$; $k \le n$; and $n = 1, \ldots N$, as

$$f(Q) = \{\mathrm{var}\,[s(j,n)]\}^{1/2} = f[Q^0]$$

$$+ \sum_{i=1}^{M} \sum_{k=1}^{n} \frac{\partial f(Q)}{\partial Q(i, n-k+1)} \bigg|_{Q^0} [Q(i, n-k+1) - Q^0(i, n-k+1)] + \eta \qquad (17)$$

in which η = the higher order terms. After neglecting the higher order terms and some algebraic manipulations, the first-order linear approximation of the nonlinear terms (shown in Appendix II) can be expressed as

$$f(Q) = \{\mathrm{var}\,[s(j,n)]\}^{1/2}$$

$$\approx \sum_{i=1}^{M} \sum_{k=1}^{n} \bar{D}(i,j,k) Q(i, n-k+1); \quad \text{for all } j \text{ and } n \ldots\ldots\ldots\ldots\ldots\ldots \ (18)$$

$$\text{where} \quad \bar{D}(i,j,k) = \frac{1}{f(Q^0)} \left\{ \left[\sum_{i=1}^{M} \sum_{k=1}^{n} \bar{A}(i,j,k)\sigma_T Q^0(i, n-k+1) \right] \bar{A}(i,j,k)\sigma_T \right.$$

$$\left. + \left[\sum_{i=1}^{M} \sum_{k=1}^{n} \bar{B}(i,j,k)\sigma_S Q^0(i, n-k+1) \right] \bar{B}(i,j,k)\sigma_S \right\} \ldots\ldots\ldots\ldots\ldots\ldots \ (19)$$

Finally, substituting Eq. 18 into Eq. 16 results in a linear approximation of deterministic equivalent of original chance-constraint

$$\sum_{i=1}^{M} \sum_{k=1}^{n} E(i,j,k) Q(i, n-k+1) \le s^*(j,n); \quad \text{for all } j \text{ and } n \ \ldots\ldots\ldots\ldots \ (20)$$

where $E(i,j,k) = \bar{\beta}(i,j,k) + F^{-1}[\alpha(j,n)]\bar{D}(i,j,k)$. If we replace the drawdown constraints in the previous deterministic management model by Eq. 20, then the model would become a stochastic one which considers the random nature of aquifer properties. The coefficient $E(i,j,k)$ in Eq. 20 can be considered as a stochastic unit response function derived from the Cooper-Jacob equation.

Solution Technique

Because the original chance-constrained management model formulation contains nonlinear terms in the drawdown constraints, a linearization procedure is performed on Eq. 16 in order to utilize the LP technique for problem solving. In the process of linearization, initial estimates of pumping rates are needed for each pumpage well during all periods, and these estimates, in turn, are used to calculate the values of each stochastic influence coefficient $E(i,j,k)$ in Eq. 20. As a result, the optimal solution obtained from the linearized management models, Eqs. 6, 20, and 8, is not necessarily the optimal solution to the original problem. An iterative procedure is required to ensure the convergence of the approximated solution to the true optimal solution.

When solving the linearized stochastic management model, the model formulation originally stated can be relaxed by dropping the demand constraints. The writer felt that, in problems of this nature, the inclusion

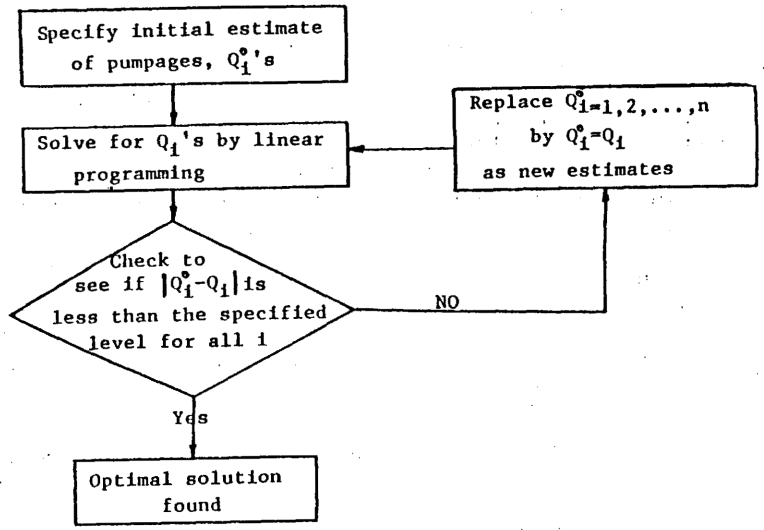

FIG. 1.—Flow Chart for Solving Linearized Chance-Constrained Groundwater Management Model

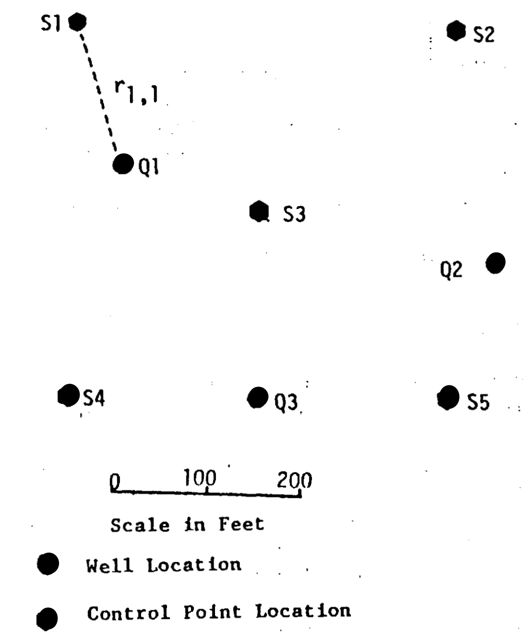

FIG. 2.—Locations of Wells and Control Points for Hypothetical Example

of a demand constraint in the management model is somewhat redundant because decision-makers generally would have some knowledge about the desirable demand level. Under a specified limitation on drawdowns, the relaxed model can solve for the maximum allowable pumpage that can be extracted from the aquifer without violating the drawdown constraints. If the maximum allowable pumpage determined by the model does not exceed the desirable demand level, then, the problem does not have a feasible solution. Decision makers would have to reconsider the drawdown limitations, performance reliabilities, or even the demand level they originally imposed on the system. On the other hand, the problem solution is obtained if the maximum allowable pumpage exceeds the demand. In other words, the relaxed model can be used by decision-makers for finding the maximum physical capacity of a groundwater system and to adjust their planning decisions accordingly.

The solution for the relaxed stochastic management model can be obtained as follows:

1. Provide an initial estimate of pumpage at each well for all periods.
2. Solve the linearized model by the linear programming technique.
3. Compare the current optimal solution of pumpage rates with the pumpage estimates from the previous iteration.
4. If the difference between solutions from two consecutive iterations is within the specified tolerance limit, stop the iteration and the optimal solution is found. Otherwise, update pumpage estimates and repeat steps 2 and 3.

A flow chart of the above solution procedures for a multi-period chance-constrained groundwater management model is shown in Fig. 1. Of course, other stopping rules, additional to the convergence criterion, can also be imposed to prevent excessive iteration during the computation. It should be noted that the global optimum to the problem, in general, cannot be guaranteed because of the nonlinear nature of the problem. Therefore, a few runs with new starting points are suggested to ensure that the overall optimum is obtained.

Model Application

Consider a hypothetical confined aquifer basin with three potential wells and five control locations where the drawdown is of interest. The locations of potential production wells and control points for this hypothetical example are predetermined and are shown in Fig. 2. From the physical layout, the distances between the production well and control point can be measured and are given in Table 1. The mean transmissivity and storage coefficient over the basin are 5,000 ft^2/day (465 m^2/day) and 0.002, respectively. The problem is to determine the optimal pumping rate for each potential well over three time periods of 50 days each, such that the resulting drawdown at each control point will not exceed a maximum allowable value with a specified reliability. The maximum allowable drawdown value at each of the five control points, $j = 1, 2, \cdots, 5$, over each period are given in Table 2. The objective function is

TABLE 1.—Distance, in ft, between Potential Pumping Wells and Control Points in Hypothetical Example

Pump wells (1)	Control Point 1 (2)	2 (3)	3 (4)	4 (5)	5 (6)
1	158	381	158	255	430
2	515	255	292	474	158
3	447	447	200	200	200

TABLE 2.—Maximum Allowable Drawdown, in ft, at each Control Point in Hypothetical Example

Period (1)	Control Point 1 (2)	2 (3)	3 (4)	4 (5)	5 (6)
1	5	5	8	5	5
2	8	8	10	8	8
3	10	10	15	10	10

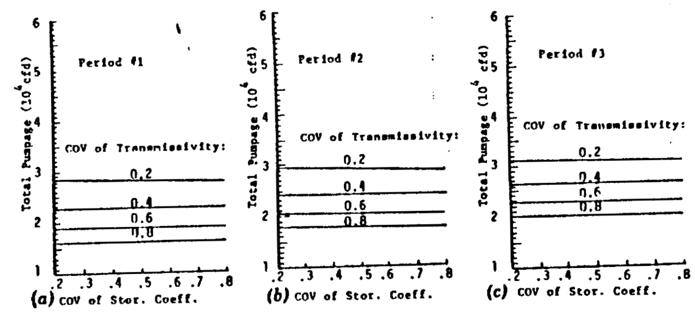

FIG. 3.—Total Allowable Pumpage from Well Field for Period No. 1 with Various Reliability Levels: (*a*) Reliability = 0.975; (*b*) Reliability = 0.950; (*c*) Reliability = 0.900

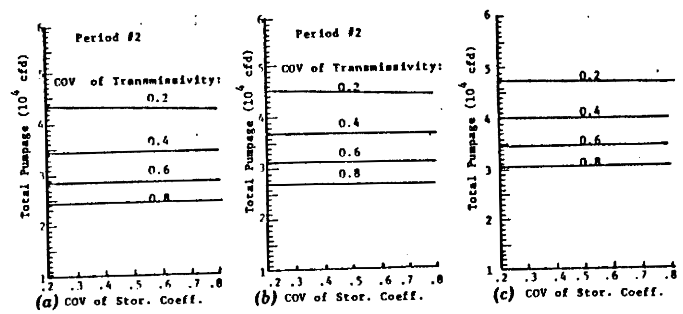

FIG. 4.—Total Allowable Pumpage from Well Field for Period No. 2 with Various Reliability Levels: (*a*) Reliability = 0.975; (*b*) Reliability = 0.950; (*c*) Reliability = 0.900

to maximize the total pumpage subject to drawdown constraints of specified reliability.

To examine the effects of the reliability level and aquifer property uncertainties on the optimal solutions, problems with different performance reliability requirements and various levels of uncertainty in aquifer transmissivity and storage coefficient are solved and the results are shown in Figs. 3–6. The uncertainty level of aquifer parameters is measured by the coefficient of variation (COV). As expected, the maximum total amount of pumpage increases as the required performance reliability level decreases. At a given required reliability level, the maximum total pumpage decreases as aquifer parameter uncertainty increases. Model results are quite insensitive to the uncertainty of the storage coefficient under a specified performance reliability level and a given uncertainty level of transmissivity. This relative insensitivity of the uncertainty level of the storage coefficient on the model results can be explained from Eq. 26 in Appendix I and Eq. 14. That is, the contribution of the uncertainty of the storage coefficient to the overall uncertainty in the total drawdown is quite insignificant. This implies that the aquifer storage coefficient could practically be treated as a constant. However, the uncertainty of the aquifer transmissivity cannot be ignored.

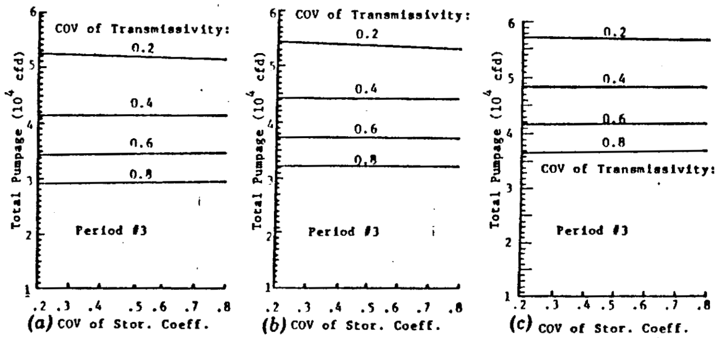

FIG. 5.—Total Allowable Pumpage from Well Field for Period No. 3 with Various Reliability Levels: (*a*) Reliability = 0.975; (*b*) Reliability = 0.950; (*c*) Reliability = 0.900

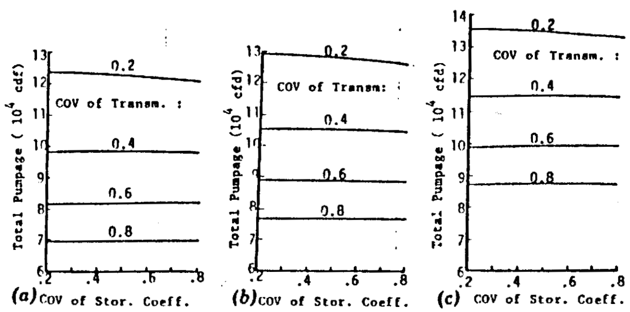

FIG. 6.—Total Allowable Pumpage from Well Field for All Three Periods with Various Reliability Levels: (*a*) Reliability = 0.975; (*b*) Reliability = 0.950; (*c*) Reliability = 0.900

Post-Optimality Simulation

During the process of transforming the original chance-constrained drawdown equation to its deterministic equivalent, the probability den-

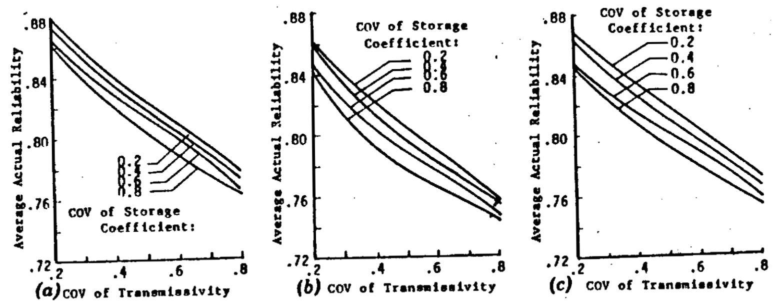

FIG. 7.—Average Actual Reliability under Various Uncertainty Levels when Model Reliability Requirement Is 90%: (*a*) Period #1; (*b*) Period #2; (*c*) Period #3

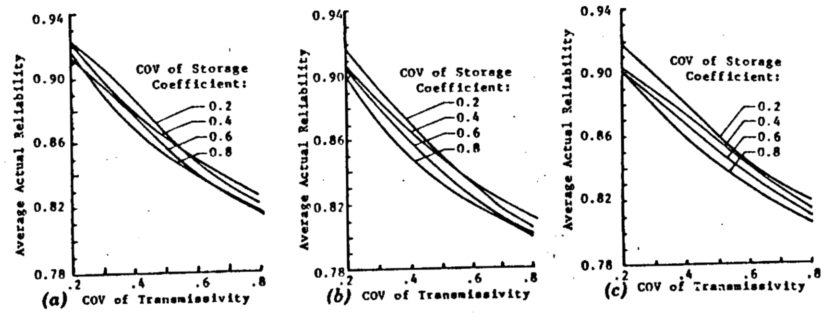

FIG. 8.—Average Actual Reliability under Various Uncertainty Levels when Model Reliability Requirement Is 95%: (*a*) Period #1; (*b*) Period #2; (*c*) Period #3

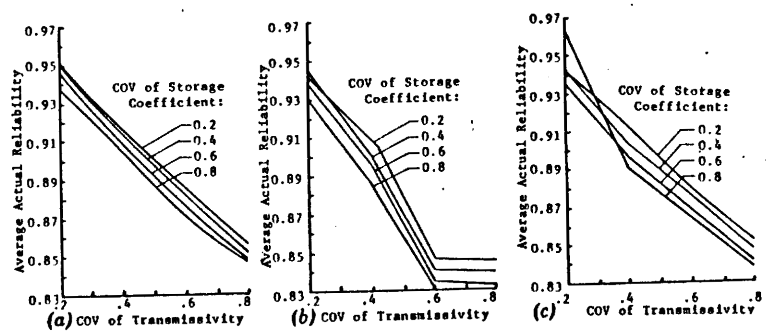

FIG. 9.—Average Actual Reliability under Various Uncertainty Levels when Model Reliability Requirement Is 97.5%: (*a*) Period #1; (*b*) Period #2; (*c*) Period #3

sity function of the random drawdown at each control point for all periods is assumed to have a normal distribution. It is worthwhile to investigate the adequacy of model results under such an assumption. In this section, a post-optimality simulation study is presented to examine how close the model outputs comply with the required system performance reliability at various control points during different planning periods. One thousand log-normally distributed independent random samples for the transmissivity and storage coefficient were generated based on their individual statistical properties used in the hypothetical example. Optimal pumpages determined from the stochastic model under various reliability requirements and uncertainty levels were used in the post-optimality simulation study. From the study, actual reliability at different control points during all planning periods were computed. It was found that actual reliability varies from one control point to another even though the required performance reliability in the model for all control points are uniformly the same. This implies, from a system operation viewpoint, that only a fraction of control points in the system which would be critical and dictate the model results. Unfortunately, identification of such critical locations in system modeling is difficult. The actual reliability averaged over a total of five control points for each period under various reliability requirements and uncertainty levels of aquifer parameters are shown in Figs. 7–9. As can be seen, the actual average reliability corresponding to the model results is lower than the specified model reliability requirements in all cases. From a practical viewpoint, the model results are acceptable when the COV of transmissivity is small. Again, the uncertainty level of the storage coefficient is not critical. This study indicated that, when the uncertainty of transmissivity is small, the use of the normality assumption to describe drawdown probability distribution is adequate. It would be interesting to investigate the adequacy of using other types of probability distribution in Eq. 19. Possible sources for such a discrepancy between actual reliability and specified reliability, especially when the COV of transmissivity is moderate or large, are the inappropriateness of the first-order used analysis in assessing statistical properties of the random drawdown and the number of potential pump wells is too small to make the CLT applicable.

Summary and Conclusions

A simple stochastic multi-period groundwater management for a homogenous, nonuniform, confined aquifer is developed. The model utilizes the concept of the unit response function that explicitly considers the random nature of aquifer properties such as the transmissivity and storage coefficient. The response function is derived from the simple Cooper-Jacob equation. The purpose of the paper is to present a methodology for formulating a simple stochastic management model applying those analytical equations for groundwater flow under idealized conditions when there is insufficient hydrogeologic information available. The use of a simple model is justified when data is lacking. However, as development progresses and more data is collected, a more sophisticated model should be employed.

Application of the model is demonstrated using a hypothetical example through which factors affecting model results are investigated. Basically, the total maximum pumpage increases as the reliability requirement and uncertainty level of aquifer properties decreases. Because the value of the storage coefficient in most confined aquifers is very small and the use of the Cooper-Jacobs equation in this study, the model results were found to be quite insensitive to its uncertainty level. However, model outputs are very sensitive to the uncertainty level of transmissivity.

In a post-optimality simulation study, it is found that the model yields rather acceptable results in complying specified reliability requirements only when uncertainty of transmissivity is small. Again, uncertainty of the storage coefficient has little effect on the compliance of required reliability. These observations could lead to the following general conclusions:

1. Effort should be given to better evaluate aquifer transmissivity including its variability. The storage coefficient in a modeling process can be treated as deterministic and its accuracy is not crucial.
2. When the uncertainty of transmissivity is moderately large, the normality assumption for random drawdown may not be appropriate. Some other types of distribution functions should be examined. Furthermore, the assessment of statistical properties of drawdown using first-order analysis may not be appropriate. There have been some investigations regarding the appropriateness of first-order analysis applied to situations where variation of system components is large (14,20).

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Appendix I.—Uncertainty Analysis of Random Drawdown

Discrete formulation of drawdown at control point j at the end of the nth period is given by Eq. 2 as

$$s(j,n) = \sum_{i=1}^{M} \sum_{k=1}^{n} \beta(i,j,k) Q(i,n-k+1)$$

where $\beta(i,j,k)$ = the unit response function, which can be derived from the Cooper-Jacob equation as

$$\beta(i,j,k) = \frac{1}{4\pi T} \{W[u(i,j,k)] - W[u(i,j,k-1)]\} \quad \text{(21)}$$

where $W[u(i,j,k)] = \ln(2.25Tt_k/r_{ij}^2 S)$. Since T and S are random variables, the unit response function, $\beta(i,j,k)$ as well as drawdown, $s(j,k)$, are both random variables because they are functions of random variables.

To estimate statistical properties of random drawdown, the first-order analysis of uncertainty is employed. Taylor's expansion of drawdown about the mean values of T and S can be expressed as

$$s(j,n) = \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{\beta}(i,j,k)Q(i,n-k+1)$$

$$+ \left.\frac{\partial s(j,n)}{\partial T}\right|_{\bar{T}} (T-\bar{T}) + \left.\frac{\partial s(j,n)}{\partial S}\right|_{\bar{S}} (S-\bar{S}) + \eta \qquad (22)$$

where $\bar{\beta}(i,j,k)$ = unit response function given in Eq. 21 in which random variables T and S are replaced by their respective mean values, $\bar{T}$ and $\bar{S}$. The first-order partial derivative of $s(j,n)$ with respect to T can be obtained as

$$\frac{\partial s(j,n)}{\partial T} = \frac{\partial}{\partial T}\left[\sum_{i=1}^{M}\sum_{k=1}^{n} \beta(i,j,k)Q(i,n-k+1)\right]$$

$$= \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{A}(i,j,k)Q(i,n-k+1) \qquad (23)$$

in which $$\bar{A}(i,j,k) = \begin{cases} \dfrac{1}{4\pi\bar{T}^2}\left[1 - \ln\left(\dfrac{2.25\bar{T}t_1}{r_{ij}^2 S}\right)\right]; & k=1 \\ \dfrac{1}{4\pi\bar{T}^2}\left[\ln\left(\dfrac{t_k}{t_{k-1}}\right)\right]; & k \ge 2 \end{cases} \qquad (24)$$

Similarly, first-order partial derivative of drawdown with respect to the storage coefficient can be obtained as

$$\frac{\partial s(j,n)}{\partial S} = \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{B}(i,j,k)Q(i,n-k+1) \qquad (25)$$

where $$\bar{B}(i,j,k) = \begin{cases} \dfrac{1}{4\pi\bar{T}\bar{S}}; & k=1 \\ 0\ ; & k \ge 2 \end{cases} \qquad (26)$$

Ignoring the higher order terms in Eq. 22, the expectation of drawdown can be approximated as Eq. 13

$$E[s(j,n)] \approx \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{\beta}(i,j,k)Q(i,n-k+1)$$

Furthermore, assuming independency of T and S the variance of drawdown can be approximated as Eq. 14

$$\text{var}[s(j,n)] \approx \left[\left.\frac{\partial s(j,n)}{\partial T}\right|_{\bar{T}}\right]^2 \sigma_T^2 + \left[\left.\frac{\partial s(j,n)}{\partial S}\right|_{\bar{S}}\right]^2 \sigma_S^2$$

$$= \left[\sum_{i=1}^{M}\sum_{k=1}^{n} \bar{A}(i,j,k)Q(i,n-k+1)\right]^2 \sigma_T^2$$

$$+ \left[\sum_{i=1}^{M}\sum_{k=1}^{n} \bar{B}(i,j,k)Q(i,n-k+1)\right]^2 \sigma_S^2$$

where σ_T and σ_S = respectively, the standard deviations of transmissivity and storage coefficient.

Appendix II.—Derivation of Eq. 18

Substituting Eq. 14 into Eq. 17, we can express $\sqrt{\text{var}[s(j,n)]}$ in terms of unknown pumpages Q's more explicitly as

$$f(Q) = \{\text{var}[s(j,n)]\}^{1/2} = \left\{\left[\sum_{i=1}^{M}\sum_{k=1}^{n} \bar{A}(i,j,k)\sigma_T Q(i,n-k+1)\right]^2\right.$$

$$\left. + \left[\sum_{i=1}^{M}\sum_{k=1}^{n} \bar{B}(i,j,k)\sigma_S Q(i,n-k+1)\right]^2\right\}^{1/2} = \{f_T^2(Q) + f_S^2(Q)\}^{1/2} \qquad (27)$$

where $f_T(Q) = \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{A}(i,j,k)\sigma_T Q(i,n-k+1)$

and $f_S(Q) = \sum_{i=1}^{M}\sum_{k=1}^{n} \bar{B}(i,j,k)\sigma_S Q(i,n-k+1)$

Eq. 17 is a first-order Taylor expansion of Eq. 27. The first terms on the right-hand side of Eq. 17, $f(Q^0)$, is the value of function $f(Q)$ calculated by using arbitrarily assumed pumpages, Q^0's. The partial derivative in the second terms of Eq. 17 can be expressed as

$$\left.\frac{\partial f(Q)}{\partial Q(i,n-k+1)}\right|_{Q^0} = \frac{1}{f(Q^0)}[f_T(Q^0)\bar{A}(i,j,k)\sigma_T + f_S(Q^0)\bar{B}(i,j,k)\sigma_S] \qquad (28)$$

Substituting Eq. 28 into Eq. 17 and multiplying it with $Q(i,n-k+1)$ and $Q^0(i,n-k+1)$, respectively, we obtain

$$f(Q) = f(Q^0) - \frac{1}{f(Q^0)}\sum_i\sum_k [f_T(Q^0)\bar{A}(i,j,k)\sigma_T$$

$$+ f_S(Q^0)\bar{B}(i,j,k)\sigma_S]Q^0(i,n-k+1) + \frac{1}{f(Q^0)}\sum_i\sum_k [f_T(Q^0)\bar{A}(i,j,k)\sigma_T$$

$$+ f_S(Q^0)\bar{B}(i,j,k)\sigma_S]Q(i,n-k+1) + \eta \qquad (29)$$

Since $f_T(Q^0)$ and $f_S(Q^0)$ are constants, they can be moved out the double summation in the second terms on the right-hand side of Eq. 29. As a result, the first and second terms cancel each other. By dropping the higher order term, η, Eq. 29 can be rewritten as Eq. 18.

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